Spectra of Quantum Algebras

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Quantum Groups Seminar
24 June 2024

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For a noncommutative algebra A,

$\operatorname{Prim} A :=$ { primitive ideals of A } $\approx$ a NC affine variety

$\operatorname{Spec} A :=$ { prime ideals of A } $\approx$ a NC affine scheme.

Equip both with Zariski topologies.

Defn. Multiparameter quantum affine n-space.

Let $\mathbf{q} = (q_{ij}) \in M_n(K^*)$ with $q_{ji} = q_{ij}^{-1}$ and $q_{ii} = 1$ for all i, j.

$$\mathcal{O}_{\mathbf{q}}(K^n) := K\langle\, x_1, \ldots, x_n \mid x_i x_j = q_{ij} x_j x_i \ \forall\ i, j \,\rangle.$$

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Single-parameter version: $q_{ij} =$ fixed $q \in K^*$ for all $i < j$;

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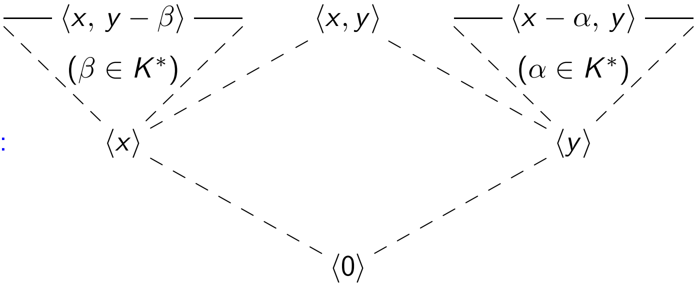
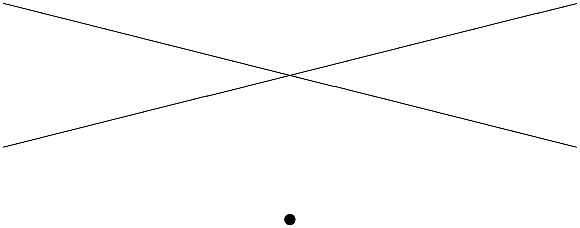
Spec A :



Prim A :



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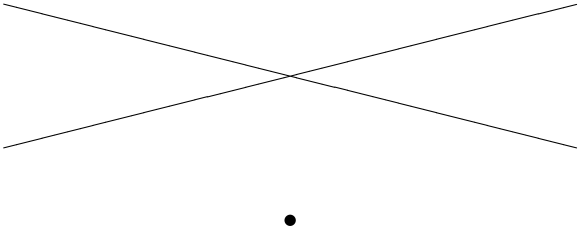

Thm. [Letzter-KG, 2000] $A = \mathcal{O}_{\mathbf{q}}(K^n)$. Assume $-1 \notin \langle q_{ij} \rangle$ or char $K = 2$. Then Prim A is a topological quotient of K^n and Spec A is a (compatible) topological quotient of Spec $\mathcal{O}(K^n)$.

These statements hold more generally for cocycle twists of commutative affine algebras graded by torsionfree abelian groups, and hence for quantum toric varieties.

Conjecture 1: Let $A =$ a generic quantized coordinate ring for an affine variety V. Then $\operatorname{Spec} A$ and $\operatorname{Prim} A$ are compatible topological quotients of $\operatorname{Spec} \mathcal{O}(V)$ and $\max \mathcal{O}(V) \approx V$.

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Defn. Quantum 2×2 matrix algebra $:=$ K-alg. $\mathcal{O}_q(M_2(K))$ with generators a, b, c, d and relations

$$ab = qba \qquad ac = qca \qquad bc = cb$$

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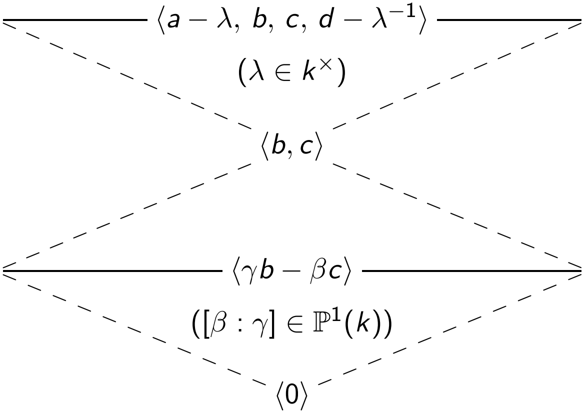
Spec A :



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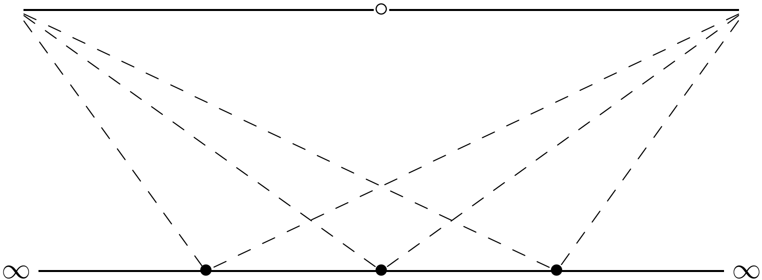

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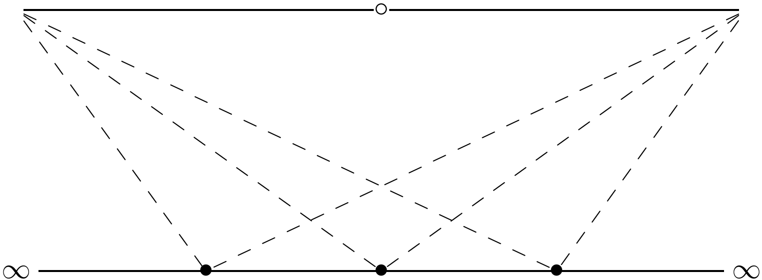

$\exists$ topological quotient map $SL_2(K) \longrightarrow$ Prim $\mathcal{O}_q(SL_2(K))$:

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \longmapsto \begin{cases} \langle a-\alpha,\ b,\ c,\ d-\delta \rangle & (\beta=\gamma=0) \\ \langle \gamma b - \beta c \rangle & (\beta,\gamma \text{ not both } =0) \end{cases}$$

Defns. $\mathcal{A}$ = a torsionfree $K[t^{\pm 1}]$-algebra such that $\mathcal{A}/(t-1)\mathcal{A}$ is commutative;

$\sim$ flat family of K-algebras $\big(\mathcal{A}/(t-q)\mathcal{A}\big)_{q\in K^*}$.

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All commutators $[a, b]$ $(= ab - ba)$ in $\mathcal{A}$ are divisible by $t - 1$.

$\therefore$ have a Lie bracket $\dfrac{1}{t-1}[-,-]$ on $\mathcal{A}$,

which induces a Lie bracket $\{-,-\}$ on $\mathcal{A}/(t-1)\mathcal{A}$.

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Additionally, there is a "product rule":

$\{a,xy\} = \{a,x\}y + x\{a,y\}$ $\forall\ a,x,y \in \mathcal{A}/(t-1)\mathcal{A}$.

$\{-,-\}$ is a "Poisson bracket".

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$$\{a,b\} = ab \qquad \{c,d\} = cd$$
$$\{a,c\} = ac \qquad \{b,d\} = bd$$
$$\{b,c\} = 0 \qquad \{a,d\} = 2bc$$

Defns. A Poisson algebra is an algebra R equipped with a Lie algebra bracket $\{-,-\} : R \times R \to R$ such that

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The Poisson-prime and Poisson-primitive spectra of R are

$\operatorname{Pspec} R := \{$ Poisson-prime ideals of R $\}$

$\operatorname{Pprim} R := \{$ Poisson-primitive ideals of R $\}$ $\subseteq$ $\operatorname{Pspec} R$,

both with Zariski topologies.

Thm. [KG, 1997] $R =$ a commutative noetherian Poisson algebra.

- Pspec R is a topological quotient of Spec R, via the map $\pi : P \longmapsto$ (largest Poisson ideal $\subseteq P$).

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- $\text{Pspec}\, R$ is a topological quotient of $\text{Spec}\, R$, via the map $\pi : P \longmapsto$ (largest Poisson ideal $\subseteq P$).

- Assume R is affine over K and satisfies the Poisson Dixmier-Moeglin Equivalence; specifically: all $P \in \text{Pprim}\, R$ are locally closed points in $\text{Pspec}\, R$.

Then $\text{Pprim}\, R$ is a topological quotient of $\max R$, via π.

Conjecture 2: Let $A =$ a generic quantized coordinate ring for an affine variety V, with semiclassical limit $\mathcal{O}(V)$.

Then $\exists$ compatible homeomorphisms $\operatorname{Spec} A \longrightarrow \operatorname{Pspec} \mathcal{O}(V)$ and $\operatorname{Prim} A \longrightarrow \operatorname{Pprim} \mathcal{O}(V)$.

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Known cases:

• Quantum affine spaces and quantum affine toric varieties [Oh-Park-Shin, 2002; Letzter-KG, 2009]

• Quantum symplectic and euclidean spaces [Oh, 2008; Oh-Park, 2002, 2010]

• $\mathcal{O}_q(SL_2(K))$ and $\mathcal{O}_q(GL_2(K))$ [KG, 2010]

• $\mathcal{O}_q(SL_3(K))$ [Fryer, 2017]

Conjecture 3: Let $\left(A_q\right)_{q \in K^*}$ = a flat family of quantized coordinate rings for an affine variety V.

Then Spec $A_p \approx$ Spec A_q and Prim $A_p \approx$ Prim A_q $\forall$ generic $p, q \in K^*$.

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Defn. A partially ordered set X is catenary iff $\forall\ x, y \in X$, all saturated chains

$$x = x_0 \lneqq x_1 \lneqq \cdots \lneqq x_n = y$$

have the same length.

Catenarity in Spec A :

- $\mathcal{O}_{\mathbf{q}}(K^n)$, $\mathcal{O}_q(SL_n(K))$, $\mathcal{O}_q(GL_n(K))$ [Lenagan-KG, 1996]
- quantized Weyl algebras [Lenagan-KG, 1996; Oh, 1997]
- quantum symplectic and euclidean spaces [Oh, 1997; Horton, 2003]
- $\mathcal{O}_q(M_{m,n}(K))$ [Cauchon, 2003]
- quantum semisimple groups [Zhang-KG, 2007; Yakimov, 2014]
- quantum Grassmannians [Launois-Lenagan-Rigal, 2008]
- quantum Schubert cells [Yakimov, 2013]
- quantum nilpotent algebras [Launois-KG, 2020]

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Catenarity in Pspec R :

- Poisson nilpotent algebras [Launois-KG, 2022]
- E.g., semiclassical limits of above A.

Stratification. [Letzter-KG, 2000; Stafford-KG, 2000]

A = a noetherian K-algebra satisfying the NC Nullstellensatz,

H = a torus $(K^*)^r$ acting rationally on A, with H-Spec A finite.

Then $\operatorname{Spec} A = \bigsqcup_{J \in H\text{-}\operatorname{Spec} A} \operatorname{Spec}_J A$ where

$$\operatorname{Spec}_J A := \{ P \in \operatorname{Spec} A \mid \bigcap_{h \in H} h(P) = J \}$$

and $\operatorname{Spec}_J A \approx \operatorname{Spec} Z_J$ where

Z_J = center of a localization of A/J,

$\cong$ a Laurent polynomial ring over K.

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$\exists$ corresponding partition $\text{Prim}\, A = \bigsqcup_{J \in H\text{-Spec}\, A} \text{Prim}_J\, A$

and each $\text{Prim}_J\, A \approx \max Z_J$.

For $J \subset J'$ in H-Spec A, define

$$\phi^{s}_{JJ'} : \mathcal{C}l(\mathrm{Spec}_J\, A) \longrightarrow \mathcal{C}l(\mathrm{Spec}_{J'}\, A), \qquad Y \longmapsto \overline{Y} \cap \mathrm{Spec}_{J'}\, A$$

$$\phi^{p}_{JJ'} : \mathcal{C}l(\mathrm{Prim}_J\, A) \longrightarrow \mathcal{C}l(\mathrm{Prim}_{J'}\, A), \qquad Y \longmapsto \overline{Y} \cap \mathrm{Prim}_{J'}\, A$$

where $\mathcal{C}l(T) = \{$ closed subsets of $T\}$.

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The closed subsets of Spec A are the subsets X such that

- $X \cap \mathrm{Spec}_J A \in \mathcal{C}l(\mathrm{Spec}_J A)$ for all J;
- $\phi^s_{JJ'}(X \cap \mathrm{Spec}_J A) \subseteq X \cap \mathrm{Spec}_{J'} A$ for all $J \subset J'$.

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Similarly for the closed subsets of Prim A.

Conjecture 4: [Brown-KG, 2015] (A as in Stratification Thm.)

For $J \subset J'$ in H-Spec A, $\exists$ an affine variety $V_{JJ'}$ and morphisms

$$\begin{array}{ccc} \mathrm{Prim}_{J'} A & & \\ & \searrow^{f_{JJ'}} & \\ \mathrm{Prim}_J A & \xrightarrow{g_{JJ'}} & V_{JJ'} \end{array}$$

such that $\phi^{p}_{JJ'}(Y) = f^{-1}_{JJ'}(\overline{g_{JJ'}(Y)})$ for $Y \in \mathcal{C}l(\mathrm{Prim}_J A)$.

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Similarly for $\phi^{s}_{JJ'}$, with morphisms of affine schemes.

Known cases: • $\mathcal{O}_q(M_2(K))$, $\mathcal{O}_q(GL_2(K))$, $\mathcal{O}_q(SL_2(K))$, $\mathcal{O}_q(SL_3(K))$ [Brown-KG, 2015]

• Poisson analog for $\mathcal{O}(SL_3(K))$ [Fryer, 2017]

E.G. Types of auxiliary data for $\operatorname{Prim} \mathcal{O}_q(GL_2(K))$:

$$\begin{array}{ccc} K^* & & \\ & \searrow^{(0,-)} & \\ (K^*)^2 & \xrightarrow{\text{incl}} & K \times K^* \end{array} \qquad\qquad \begin{array}{ccc} (K^*)^2 & & \\ & \searrow^{\text{mult}} & \\ (K^*)^2 & \xrightarrow{\text{pr}_2} & K^* \end{array}$$

$$\begin{array}{ccc} (K^*)^2 & & \\ & \searrow^{\text{mult}} & \\ K^* & \xrightarrow{\text{id}} & K^* \end{array}$$

Concerning

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$\phi^{p}_{JJ'}$ sends finite sets to $\varnothing$

$\therefore$ $\phi^{p}_{JJ'}$ cannnot be given by closures of images under any map $(K^*)^2 \longrightarrow K^*$,

nor by inverse images under any map $K^* \to (K^*)^2$.

THANK YOU !

E.G. $A = \mathcal{O}_q(GL_2(K))$, $q \neq \sqrt[\bullet]{1}$. Spec A :

$$\langle a-\alpha,\ b,\ c,\ d-\delta\rangle \quad (\alpha,\delta \in k^\times)$$

$$\langle b, c, p_{\alpha,\delta}\rangle \quad (\alpha,\delta \in k^\times)$$

$$\langle b, c\rangle$$

$$\langle b, ad-\lambda\rangle \quad (\lambda \in k^\times)$$

$$\langle b\rangle$$

$$\langle b-\mu c,\ D_q-\lambda\rangle \quad (\mu,\lambda \in k^\times)$$

$$\langle p'_{\mu,\lambda}\rangle \quad (\mu,\lambda \in k^\times)$$

$$\langle 0\rangle$$

$$\langle c, ad-\lambda\rangle \quad (\lambda \in k^\times)$$

$$\langle c\rangle$$